A packet switch with a priority scheduling discipline: performance analysis

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Abstract

In this paper, we analyze a discrete-time GI-G_1, Geo_2-1 preemptive resume priority queue. We consider two classes of packets which have to be served, where one class has preemptive resume priority over the other. The high-priority class contains packets with generally distributed service times while the low-priority packets are assumed to have geometrically distributed service times. We show that the use of generating functions is beneficial for analyzing the system contents and packet delay of both classes. Performance measures of system contents and packet delay are calculated. We apply these theoretical results on the special case of a packet switch.

Keywords: preemptive resume priority, queueing, generating functions

In recent years, there has been much interest devoted to incorporating multimedia applications in packet switching networks. Different types of traffic need different Quality of Service (QoS) standards, but share the same network resources, such as buffers and bandwidth. For real-time applications, it is important that mean delay and delay-jitter are bound, while for non real-time applications such as data transfer, the Loss Ratio (LR) often is the restrictive quantity. In general, one can distinguish two priority strategies, which will be referred to as Loss priority and Delay priority.

Loss priority schemes attempt to minimize the packet loss of loss-sensitive traffic (such as data). An overview and classification of some Loss priority (or discarding) strategies can be found in Cidon (1994), Liu (1997) and Gelenbe (2001).

Delay priority schemes attempt to guarantee acceptable delay boundaries to delay-sensitive traffic (such as voice/video). Amongst these Delay Priority schemes are some well-known strategies like weighted round-robin (WRR), weighted fair queueing (WFQ) or generalized processor sharing (GPS), earliest deadline first (EDF), probabilistic priority (PP) and (strict) priority scheduling. In a queueing system with WRR (see e.g. Liu (1997) and references therein), WFQ or GPS (see e.g. Parekh (1994) and references), the server serves a number of queues by a weighted schedule. Delay sensitive traffic is assigned a higher weight, i.e., (in average) delay-sensitive traffic is more served than delay-insensitive traffic. When the EDF scheduling is applied, deadlines are imposed on the packets that have to be served (based on their QoS constraints) and packets are transmitted in the order of their deadlines (see Liebeherr (1999) and references therein). A PP scheduling discipline (see e.g. Tham (2002)) serves a given number of priority queues in a probabilistic manner. Each priority queue is assigned a parameter p_i, which determines the probability that a packet from that priority queue is served next. All these scheduling disciplines try to give some kind of priority to delay-sensitive traffic over delay-insensitive traffic. The most drastic in this respect is the strict priority scheduling. With this scheduling, as long as delay-sensitive (or high-priority) packets are present in the queueing system, this type of traffic is served. Delay-insensitive packets can thus only be transmitted when no delay-sensitive traffic is present in the system. As already mentioned, this is the most drastic way to meet the QoS constraints of delay-sensitive traffic (and thus the scheduling with the most disadvantageous consequences on the delay-insensitive traffic), but also the easiest to implement. (see e.g. , Morgan (1991) and the references therein).

In this paper, we will focus on the effect of a specific Delay priority scheme, i.e., the preemptive resume priority scheduling discipline. We assume that delay-sensitive traffic has (strict) preemptive priority over delay-insensitive traffic, i.e., when the server becomes idle, a packet of delay-sensitive traffic, when available, will always be scheduled next. In the remaining, we will refer to the delay-sensitive and delay-insensitive traffic as high- and low-priority traffic respectively. Newly arriving high-priority traffic interrupts transmission of a low-priority packet that has already commenced, and the interrupted low-priority packet can resume its transmission when all the high-priority traffic has left the system.

In the literature, there have been a number of contributions with respect to strict priority scheduling. An overview of some basic strict priority queueing models can be found in Jaiswal (1968), Miller (1960), Takacs (1964) and Takagi (1991) and the references therein. Khamisy (1992), Laevens (1998), Takine (1994b) and Walraevens (2003) have studied discrete-time HOL priority queues with deterministic service times equal to one slot. Khamisy (1992) analyzes the system contents for the different classes, for a queue fed by a two-state Markov modulated arrival process. Laevens (1998) analyzes the system contents and cell delay in the case of a multiserver queue. In Takine (1994b), the system contents and the delay for Markov modulated high-priority arrivals and geometrically distributed low-priority arrivals are presented. Walraevens (2003) studies the system contents and cell delay, in the special case of an output queueing switch with Bernoulli arrivals. All these models have a packet transmission time of a single slot in common. Furthermore, preemptive resume priority queues have been analyzed in Machihara (1995), Takine (1994a) and Walraevens (2004). Machihara (1995) analyzes waiting times when high-priority arrivals are generated according to a MAP process. Takine (1994a) studies the waiting times of customers arriving to a queue according to independent MAP processes. Finally, Walraevens (2000) analyzes system contents and packet delay when service times are geometrically distributed.

In this paper, we analyze the system contents and packet delay of high- and low-priority traffic in a discrete-time single-server buffer for a preemptive resume priority scheme and per-slot i.i.d. arrivals. The arrivals of the different types of packets are not mutually independent however, i.e., the analyzed model accounts for a possible correlation between the number of arrivals of both classes in one slot. Therefore the different classes can not be analyzed separately (i.e., as a model with server interruptions for the low-priority packets, see e.g. Fiems (2002)), which complicates the analysis. The transmission times of the high-priority packets are assumed to be generally distributed, while the transmission times of the low-priority packets are assumed to be geometrically distributed. The latter assumption is mainly made to make the analysis tractable.

We will use the results obtained in this paper to analyze non-blocking packet switches with output queues. Packet switches can mainly be categorized in 3 categories, namely switches with input queues, output queues or shared queues. In the first category, contending packets are queued at the inlets of the switch. Because of this, Head-Of-Line (HOL) blocking can occur, i.e., a packet can not be send to its designated outlet (although this outlet could be available), if another packet is blocking the same inlet. Switches with output queues do not suffer from HOL blocking, but the architecture of the latter ones is generally more complex. In switches with shared buffering all packets that have to be queued are put in one buffer. Overview of architectures of all these types of switches can be found e.g. in Karol (1987), Pattavina (1993a), Pattavina (1993b) and Guizani (2001). Most of these architectures are proposed and analyzed in an ATM-context. The main advantage of ATM switches is that the cells are all of the same size, and thus discrete-time queueing models with a single-slot service time are very appropriate for analyzing these types of switches. Examples can be found in the book of Bruneel and Kim (1993a). For nowadays packet switching networks on the other hand, the length of the packets are not all of the same size and thus we need queueing models with more generally distributed service times, which are generally more difficult to analyze.

We will demonstrate that an analysis based on generating functions is extremely suitable for modelling this type of buffers with a priority scheduling discipline, with correlation between the arrivals of the different priority classes, and with service times of more than one slot. From these generating functions, expressions for some interesting performance measures - such as means, variances and approximate tail probabilities of system contents and packet delay - can be calculated. Determining the tail behavior of the system contents and packet delay is one of the main contributions of the paper. Although these are important quantities in the evaluation of the QoS of high- and low-priority packet streams, this has received only little attention up till now, because the derivation of these quantities is far from straightforward. Indeed, as will be shown later on, the distribution of the system contents and packet delay of low-priority packets not necessarily has an exponential asymptotic behavior.

The remainder of this paper is structured as follows. In the following section, we present the mathematical model. In sections 2 and 3, we will then analyze the steady-state system contents and packet delay of both classes. In section 4, we give expressions for some moments of the system contents and packet delay of both classes, while we demonstrate in section 5 how to calculate approximate tail probabilities of these stochastic variables. The application of these results in the special case of a packet switch is treated in section 6 and some conclusions are formulated in section 7.

1 Mathematical model

We consider a discrete-time single-server system with infinite buffer space. Time is assumed to be slotted. There are two types of packets arriving to the system, which will be referred to as packets of class-1 (high-priority) and packets of class-2 (low-priority) respectively. The number of arrivals of class-j during slot k are i.i.d. and are denoted by $a_{j,k}$ $(j = 1, 2)$. The joint probability generating function (pgf) of $a_{1,k}$ and $a_{2,k}$ is defined as $A(z_1, z_2) \triangleq \mathrm{E}[z_1^{a_{1,k}} z_2^{a_{2,k}}]$. Note that the number of arrivals of both classes can be correlated during one slot. The marginal pgf's of the number of arrivals of class-j during a slot are denoted by $A_j(z)$ $(j = 1, 2)$ and are given by $A(z, 1)$ and $A(1, z)$ respectively. We will furthermore denote the mean arrival rate of class-j packets during a slot by $\lambda_j \triangleq \mathrm{E}[a_{j,k}] = A_j'(1)$ $(j = 1, 2)$.

The service times of class-1 packets are i.i.d. and generally distributed. Their pgf is denoted by $S_1(z)$. The service times of class-2 packets are assumed to be i.i.d. and geometrically distributed with parameter β_2. Their pgf is thus given by $S_2(z) = \dfrac{(1-\beta_2)z}{1-\beta_2 z}$. In the remainder, we denote $\bar{\beta}_2$ as $1 - \beta_2$.

The class-1 packets are assumed to have preemptive resume priority over the class-2 packets and within one class the scheduling is FCFS. The mean service time of a class-j packet, i.e., the mean time a class-j packet stays in the server is given by μ_j $(j = 1, 2)$. Finally, the load offered by class-j packets is given by $\rho_j \triangleq \lambda_j \mu_j$. The total load is then given by $\rho_T \triangleq \rho_1 + \rho_2$. We assume a stable system, i.e., $\rho_T < 1$.

2 System contents

In this subsection, we analyze the steady-state system contents, i.e., the number of packets in the system of both classes. As opposed to analyses that model the priority structure as service interruptions, we will analyze the *joint* pgf of the system contents of both classes, whereas with a model with service interruptions the system contents of both classes are calculated separately. From the joint pgf, we can easily obtain the marginal pgf's of the system contents of both classes and the pgf of the total system contents, as will be shown at the end of this section. We denote the system contents of class-1 packets and class-2 packets at the beginning of slot k by $u_{1,k}$ and $u_{2,k}$ respectively. Their joint pgf is defined as

$$U_k(z_1, z_2) \triangleq \mathrm{E}\left[z_1^{u_{1,k}} z_2^{u_{2,k}}\right].$$

Since service times of class-1 packets are generally distributed, the set $\{u_{1,k}, u_{2,k}\}$ does not form a Markov chain. Therefore, we introduce a new stochastic variable $r_{1,k}$ as follows: $r_{1,k}$

indicates the remaining number of slots needed to transmit the class-1 packet in service at the beginning of slot k, if $u_{1,k} > 0$, and $r_{1,k} = 0$ if $u_{1,k} = 0$. $\{r_{1,k}, u_{1,k}, u_{2,k}\}$ constitutes a Markovian state description of the system at the beginning of slot k, since the same random variables at the beginning of the next slot can be fully characterized by the random variables at the beginning of slot k (as will be shown by the system equations). Note that since the class-2 service times are geometrically distributed, defining a residual service time for these class-2 packets is not necessary, since this residual service time is always characterized by the same (geometric) distribution (independently of how many slots the packet has already received service). If $s^*_{1,k}$ indicates the service time of the next class-1 packet to receive service at the beginning of slot k, the following system equations can be established:

<table>
<tr><td>If:</td><td>$\begin{cases} r_{1,k} = 0 \\ u_{2,k} = 0 \end{cases}$</td><td>$\begin{cases} r_{1,k} = 0 \\ u_{2,k} > 0 \end{cases}$</td><td>$r_{1,k} = 1$</td><td>$r_{1,k} > 1$</td></tr>
<tr><td>$u_{1,k+1} =$</td><td colspan="2">$a_{1,k}$</td><td>$u_{1,k} - 1 + a_{1,k}$</td><td>$u_{1,k} + a_{1,k}$</td></tr>
<tr><td>$u_{2,k+1} =$</td><td>$a_{2,k}$</td><td>$\begin{cases} u_{2,k} + a_{2,k} & \text{with prob. } \beta_2 \\ u_{2,k} - 1 + a_{2,k} & \text{with prob. } \bar{\beta}_2 \end{cases}$</td><td colspan="2">$u_{2,k} + a_{2,k}$</td></tr>
<tr><td>$r_{1,k+1} =$</td><td colspan="3">$\begin{cases} 0 & \text{if } u_{1,k+1} = 0 \\ s^*_{1,k} & \text{if } u_{1,k+1} > 0 \end{cases}$</td><td>$r_{1,k} - 1$</td></tr>
</table>

This table can be explained as follows: a class-2 packet can be served when there are no class-1 packets in the system (i.e., when $r_{1,k} = 0$). Since the service times of the class-2 packets are geometric, a packet in service leaves the server after that slot with probability $\bar{\beta}_2$. If it needs at least another slot service time (with probability β_2), it only stays in the server when no new class-1 packets arrive. If new class-1 packets arrive the service of the class-2 packet in service is preempted and the packet has to go back to the queue. When a class-1 packet is in service, it leaves the system when $r_{1,k} = 1$, and needs at least another slot service time when $r_{1,k} > 1$. In the latter case, its residual service time is decreased by one.

Now, let us define $P_k(x_1, z_1, z_2)$ as the joint pgf of the state vector $(r_{1,k}, u_{1,k}, u_{2,k})$:

$$P_k(x_1, z_1, z_2) \triangleq \mathrm{E}[x_1^{r_{1,k}} z_1^{u_{1,k}} z_2^{u_{2,k}}].$$

Using the system equations, we can derive a relation between $P_k(x_1, z_1, z_2)$ and $P_{k+1}(x_1, z_1, z_2)$. In the remainder we define $\mathrm{E}[X\{Y\}]$ as $\mathrm{E}[X|Y]$ $\mathrm{Prob}[Y]$. Taking into account the statistical independence of the random variables $(r_{1,k}, u_{1,k}, u_{2,k})$ and $(a_{1,k}, a_{2,k})$, we find:

(1)
$$\begin{aligned}
&P_{k+1}(x_1, z_1, z_2) = \\
&A(0, z_2)P_k(0, 0, 0) + (A(z_1, z_2) - A(0, z_2))S_1(x_1)P_k(0, 0, 0) \\
&+A(0, z_2)(\beta_2 + \frac{\bar{\beta}_2}{z_2})(P_k(0, 0, z_2) - P_k(0, 0, 0)) \\
&+(A(z_1, z_2) - A(0, z_2))S_1(x_1)(\beta_2 + \frac{\bar{\beta}_2}{z_2})(P_k(0, 0, z_2) - P_k(0, 0, 0)) \\
&+A(0, z_2)\mathrm{E}[z_2^{u_{2,k}}\{r_{1,k} = u_{1,k} = 1\}] + \frac{1}{z_1}A(z_1, z_2)S_1(x_1)\mathrm{E}[z_1^{u_{1,k}} z_2^{u_{2,k}}\{r_{1,k} = 1\}] \\
&-A(0, z_2)S_1(x_1)\mathrm{E}[z_2^{u_{2,k}}\{r_{1,k} = u_{1,k} = 1\}] \\
&+\frac{1}{x_1}A(z_1, z_2)\{P_k(x_1, z_1, z_2) - x_1\mathrm{E}[z_1^{u_{1,k}} z_2^{u_{2,k}}\{r_{1,k} = 1\}] - P_k(0, 0, z_2)\}.
\end{aligned}$$

We assume that the system is stable and as a result $P_k(x_1, z_1, z_2)$ and $P_{k+1}(x_1, z_1, z_2)$ converge both to a common steady-state value $P(x_1, z_1, z_2)$. By taking the $k \to \infty$ limit of equation (1), we obtain:

(2)
$$\begin{aligned}
&[x_1 - A(z_1, z_2)]P(x_1, z_1, z_2) = \\
&x_1A(0, z_2)(1 - S_1(x_1))\left(\bar{\beta}_2 - \frac{\bar{\beta}_2}{z_2}\right)P(0, 0, 0) \\
&+x_1A(z_1, z_2)S_1(x_1)\left(\bar{\beta}_2 - \frac{\bar{\beta}_2}{z_2}\right)P(0, 0, 0) \\
&+x_1A(0, z_2)(1 - S_1(x_1))\left(\beta_2 + \frac{\bar{\beta}_2}{z_2}\right)P(0, 0, z_2) \\
&+A(z_1, z_2)\left(x_1S_1(x_1)\left(\beta_2 + \frac{\bar{\beta}_2}{z_2}\right) - 1\right)P(0, 0, z_2) \\
&+x_1A(0, z_2)(1 - S_1(x_1))R(0, z_2)
\end{aligned}$$

$$+x_1 A(z_1,z_2)(S_1(x_1)-z_1)R(z_1,z_2),$$

where the function $R(z_1,z_2) \triangleq \lim_{k\to\infty} \frac{1}{z_1}\mathrm{E}\left[z_1^{u_{1,k}} z_2^{u_{2,k}}\{r_{1,k}=1\}\right]$. It now remains for us to determine the unknown functions $P(0,0,z_2)$, $R(0,z_2)$, $R(z_1,z_2)$ and the unknown parameter $P(0,0,0)$ in equation (2). This can be done in a few steps. First, we observe that $P(x_1,0,z_2)=P(0,0,z_2)$ for all x_1 and z_2, due to the fact that $r_{1,k}=0$ if and only if $u_{1,k}=0$. If we put $z_1=0$ and use this property in equation (2), we obtain:

$$\begin{aligned} P(0,0,z_2) &= A(0,z_2)\left[\left(\bar{\beta}_2-\frac{\bar{\beta}_2}{z_2}\right)P(0,0,0)\right.\\ &+ \left.\left(\beta_2+\frac{\bar{\beta}_2}{z_2}\right)P(0,0,z_2)+R(0,z_2)\right]. \end{aligned} \tag{3}$$

Next, we notice that the function $P(x_1,z_1,z_2)$ must be bounded for all values of x_1 and z_j such that $|x_1|\le 1$ and $|z_j|\le 1$ $(j=1,2)$ since $P(x_1,z_1,z_2)$ is a pgf. In particular, this should be true for $x_1=A(z_1,z_2)$ and $|z_j|\le 1$ $(j=1,2)$, since $|A(z_1,z_2)|\le 1$ for all such $|z_j|$, because $A(z_1,z_2)$ is a pgf. The above implies that if we choose $x_1=A(z_1,z_2)$ in equation (2), where $|z_j|\le 1$, the left hand side of this equation vanishes. The same must then be true for the right hand side of equation (2), which yields the following relation for $R(z_1,z_2)$:

$$\begin{aligned} &A(z_1,z_2)(z_1-S_1(A(z_1,z_2)))R(z_1,z_2)=\\ &A(z_1,z_2)S_1(A(z_1,z_2))\left(\bar{\beta}_2-\frac{\bar{\beta}_2}{z_2}\right)P(0,0,0)\\ &+S_1(A(z_1,z_2))\left(A(z_1,z_2)\left(\beta_2+\frac{\bar{\beta}_2}{z_2}\right)-1\right)P(0,0,z_2), \end{aligned} \tag{4}$$

where we have used equation (3) to eliminate $R(0,z_2)$. Next, we notice that $A(z_1,z_2)R(z_1,z_2)$ must be bounded for all values of z_j such that $|z_j|\le 1$ $(j=1,2)$. In particular, this should be true for $z_1=Y(z_2)$, with $Y(z_2)\triangleq S_1(A(Y(z_2),z_2))$ and $|z_2|\le 1$, since it follows by Rouché's theorem that there is exactly one solution for $Y(z_2)$, with $|Y(z_2)|\le 1$ for all such z_2. Notice that $Y(1)$ equals 1. The above implies that if we choose $z_1=Y(z_2)$ in equation (4), where

$|z_2| \leq 1$, the left hand side of this equation vanishes. The same must then be true for the right hand side, yielding

$$P(0,0,z_2) = \frac{A(Y(z_2),z_2)\bar{\beta}_2(z_2-1)}{z_2 - A(Y(z_2),z_2)(\bar{\beta}_2+\beta_2 z_2)} P(0,0,0). \tag{5}$$

Using this equation and equations (3)-(4) in equation (2), an almost fully determined version for $P(x_1, z_1, z_2)$ can be derived:

$$\begin{aligned} P(x_1,z_1,z_2) = & \frac{P(0,0,0)\bar{\beta}_2(z_2-1)}{z_2 - A(Y(z_2),z_2)(\bar{\beta}_2+\beta_2 z_2)} \Big[A(Y(z_2),z_2) \\ & - \frac{x_1 z_1 (A(Y(z_2),z_2) - A(z_1,z_2))(S_1(x_1) - S_1(A(z_1,z_2)))}{(x_1 - A(z_1,z_2))(z_1 - S_1(A(z_1,z_2)))} \Big]. \end{aligned} \tag{6}$$

Finally, in order to find an expression for $P(0,0,0)$, we put $x_1 = z_1 = z_2 = 1$ and use de l'Hopital's rule in equation (6). Therefore we need to find the value of $Y'(1)$. By taking the derivative of both sides of the definition $Y(z_2) = S_1(A(Y(z_2),z_2))$ for $z_2 = 1$, we obtain $Y'(1) = \lambda_2 \mu_1/(1-\rho_1)$. Using this expression, we find the expected result $P(0,0,0) = 1 - \rho_T$.

From (6), some useful pgf's can be calculated. First, we can calculate the joint steady-state pgf of the system contents of class-1 packets and the residual service time of the class-1 packet in service:

$$\begin{aligned} P_1(x,z) & \triangleq \lim_{k\to\infty} \mathrm{E}\left[x^{r_{1,k}} z^{u_{1,k}}\right] = P(x,z,1) \\ & = (1-\rho_1)\left[1 - xz\frac{(1-A_1(z))(S_1(x) - S_1(A_1(z)))}{(x - A_1(z))(z - S_1(A_1(z)))}\right]. \end{aligned}$$

This joint pgf is independent of class-2 packets, due to the preemptive priority scheduling. From the point-of-view of class-1 packets it is as if they are the only packets in the system. This pgf was also derived in Bruneel (1993b), in which a single-class GI-G-1 system was analysed. More importantly, we can calculate the steady-state joint pgf of the system contents of class-1 and class-2 packets from equation (6). It is given by:

$$U(z_1,z_2) = P(1,z_1,z_2) = \frac{(1-\rho_T)\bar{\beta}_2(z_2-1)}{z_2 - A(Y(z_2),z_2)(\bar{\beta}_2+\beta_2 z_2)} \left[A(Y(z_2),z_2) - z_1\frac{(A(Y(z_2),z_2)-A(z_1,z_2))(1-S_1(A(z_1,z_2)))}{(1-A(z_1,z_2))(z_1-S_1(A(z_1,z_2)))}\right]. \tag{7}$$

If we assume $S_1(z) = \frac{(1-\beta_1)z}{1-\beta_1 z}$, i.e., the service times of class-1 packets are geometrically distributed with parameter β_1, we obtain the same result as found in a previous study Walraevens (2004), where both classes were assumed to have service times which are geometrically distributed.

From equation (7), we can derive expressions for the steady-state pgf of the system contents of class-1 packets and class-2 packets at the beginning of an arbitrary slot:

$$U_1(z) \triangleq \lim_{k\to\infty} \mathrm{E}[z^{u_{1,k}}] = U(z,1) = (1-\rho_1)\frac{S_1(A_1(z))(z-1)}{z-S_1(A_1(z))}. \tag{8}$$

$$\begin{aligned} U_2(z) &\triangleq \lim_{k\to\infty} \mathrm{E}\left[z^{u_{2,k}}\right] = U(1,z) \\ &= \frac{(1-\rho_T)\bar{\beta}_2 A_2(z)(z-1)}{z-A(Y(z),z)(\bar{\beta}_2+\beta_2 z)}\frac{A(Y(z),z)-1}{A_2(z)-1}. \end{aligned} \tag{9}$$

Furthermore we can calculate $U_T(z)$, the pgf of the total system contents from equation (7):

$$\begin{aligned} U_T(z) &\triangleq \lim_{k\to\infty} \mathrm{E}[z^{u_{1,k}+u_{2,k}}] = U(z,z) \\ &= \frac{(1-\rho_T)\bar{\beta}_2(z-1)}{z-A(Y(z),z)(\bar{\beta}_2+\beta_2 z)} \\ &\quad \left[A(Y(z),z) - z\frac{(A(Y(z),z)-A_T(z))(1-S_1(A_T(z)))}{(1-A_T(z))(z-S_1(A_T(z)))}\right]. \end{aligned} \tag{10}$$

3 Delay

The packet delay is defined as the total amount of time a packet spends in the system, i.e., the number of slots between the end of the packet's arrival slot and the end of its departure slot. We can analyze the packet delay of class-1 packets as if they are the only packets in the system. This is e.g. done in Bruneel (1993a) and the pgf of the packet delay of class-1 packets is given by

$$D_1(z) = \frac{1-\rho_1}{\lambda_1} \frac{S_1(z)(z-1)}{z - A_1(S_1(z))} \frac{1 - A_1(S_1(z))}{1 - S_1(z)}. \tag{11}$$

Because of the priority discipline, an expression for $D_2(z)$ will be a bit more involved. Because of the correlation between the number of per-slot arrivals of both classes, we cannot straightforwardly use a model with service interruptions (as done in Fiems (2002)). We will use the expression (7) of the joint pgf of the system contents of both classes and an analysis largely based on the concept of sub-busy periods, which is frequently used in queueing models in the literature in order to analyse busy periods and the likes. We tag a class-2 packet that enters the buffer during slot k. Let us refer to the packets in the system at the end of slot k, but that have to be served before the tagged packet as the "primary packets". So, basically, the tagged class-2 packet can (partly) be transmitted, when the buffer is free of primary class-2 packets and of class-1 packets. In order to analyse the delay of the tagged class-2 packet, the number of class-1 packets and class-2 packets that are served between the arrival slot of the tagged class-2 packet and its departure slot is important, not the precise order in which they are served. Therefore, in order to facilitate the analysis, we will consider an equivalent virtual system with an altered service discipline. We assume that from slot $k+1$ on, the order of service for class-1 packets (those in the queue at the end of slot k and newly arriving ones) is LCFS instead of FCFS in the equivalent system (the transmission of class-2 packets remains FCFS). So, according to the new service discipline, a primary packet can enter the server, when the system becomes free (for the first time) of class-1 packets that arrived during and after the service time of the primary packet that predecessed it. Let $v_{1,m}^{(k)}$ denote the length of the time period during which the server is occupied by the m-th class-1 packet in the system at the beginning of slot $k+1$, i.e., the time period starting at the beginning of the service of that packet and terminating when the system becomes free (for the first time) of class-1 packets which arrived during and after its service time. Analogously, let $v_{2,m}^{(k)}$ denote the length of the time period during which the server is occupied by the m-th class-2 packet in the system at the beginning of slot $k+1$. The $v_{j,m}^{(k)}$'s ($j = 1, 2$) are called sub-busy periods, caused by the m-th class-j packet in the system at the beginning of slot $k+1$. The delay of the tagged class-2 packet is thus the sum of all sub-busy periods caused by the primary packets.

As stated in Walraevens (2002) - where a system with a non-preemptive priority discipline is studied - the $v_{j,m}^{(k)}$ are i.i.d. random variables and their pgf is implicitly defined by $V_j(z) = S_j(zA_1(V_1(z)))$. The only exception is the sub-busy period of the class-1 packet which was already partly transmitted when the tagged class-2 packet arrives (if this event occurs). Notice that, since service times of class-2 packets are geometrically distributed, the sub-busy period caused by the class-2 packet which was already partly transmitted when the tagged packet arrives (if a class-2 packet is being served at that time) is also characterized by $V_2(z)$. Service times of class-1 packets are generally distributed though and as a result the sub-busy period of the (possible) class-1 packet in the server during slot k is a bit different (for detailed calculations see Walraevens (2002)). Based on these observations, the pgf $D_2(z)$ of the delay of a tagged class-2 packet can then be calculated - in a similar way as in Walraevens (2002) - and yields

$$D_2(z) = \frac{1-\rho_T}{\rho_2} \frac{z(A(V_1(z), V_2(z)) - A_1(V_1(z)))}{zA_1(V_1(z)) - A(V_1(z), V_2(z))}. \tag{12}$$

4 Calculation of moments

The functions $Y(z)$, $V_1(z)$ and $V_2(z)$ can only be explicitly found in case of some simple arrival and service processes. Their derivatives for $z = 1$, necessary to calculate the moments of the system contents and the packet delay, on the contrary, can be calculated in closed-form. For example $Y'(1)$ was already calculated in section 2 and the first derivatives of $V_j(z)$ for $z = 1$ are given by $V_j'(1) = \mu_j/(1-\rho_1)$ with $j = 1, 2$. We define $\lambda_{ij} \triangleq \left.\frac{\partial^2 A(z_1, z_2)}{\partial z_i \partial z_j}\right|_{z_1=z_2=1}$ and $\mu_{11} \triangleq \left.\frac{d^2 S_1(z)}{dz^2}\right|_{z=1}$ with $i, j = 1, 2$. Now we can calculate the mean system contents and the mean packet delay of both classes by taking the first derivatives of the respective pgf's for $z = 1$.

We find

$$\mathrm{E}[u_1] = \rho_1 + \frac{\lambda_{11}\mu_1 + \lambda_1^2\mu_{11}}{2(1-\rho_1)}, \tag{13}$$

for the mean system contents of class-1 packets and

$$\begin{aligned} \mathrm{E}[u_2] &= \rho_2 + \frac{\lambda_2\beta_2\rho_T}{(1-\beta_2)(1-\rho_T)} + \frac{\lambda_{22}}{2(1-\beta_2)(1-\rho_T)} + \frac{\lambda_{12}\mu_1}{1-\rho_T} \\ &\quad + \frac{\lambda_2(\lambda_{11}\mu_1^2 + \lambda_1\mu_{11})}{2(1-\rho_T)(1-\rho_1)}, \end{aligned} \tag{14}$$

for the mean system contents of class-2 packets. Furthermore, we find

$$\mathrm{E}[d_1] = \mu_1 + \frac{\lambda_{11}\mu_1 + \lambda_1^2\mu_{11}}{2\lambda_1(1-\rho_1)}, \tag{15}$$

for the mean packet delay of a class-1 packet and

$$\begin{aligned} \mathrm{E}[d_2] &= \mu_2 + \frac{\beta_2\rho_T}{(1-\beta_2)(1-\rho_T)} + \frac{\lambda_{22}}{2\lambda_2(1-\beta_2)(1-\rho_T)} + \frac{\lambda_{12}\mu_1}{\lambda_2(1-\rho_T)} \\ &\quad + \frac{\lambda_{11}\mu_1^2 + \lambda_1\mu_{11}}{2(1-\rho_T)(1-\rho_1)}, \end{aligned} \tag{16}$$

for the mean packet delay of a class-2 packet. Note that equations (13) - (16) satisfy Little's law $\mathrm{E}[d_j] = \mathrm{E}[u_j]/\lambda_j$ ($j = 1, 2$). Notice the appearance of λ_{12} in formula's (14) and (16). This λ_{12} gives the influence of the correlation between the number of per-slot arrivals of both classes ($\lambda_{12} = \lambda_1\lambda_2$ when these stochastic variables are not correlated).

In a similar way, expressions for the variance (and higher moments) can be calculated by taking the appropriate derivatives of the respective generating functions as well. We will show figures of the variance of the system contents and packet delay in section 6.

5 Tail behavior

Not only the moments of the system contents and packet delay are important, but also, and especially, the tail distribution of these quantities, which are often used to impose statistical bounds on the guaranteed QoS for both classes.

From the generating functions of the system contents and packet delay of class-1 and class-2 packets derived in sections 2 and 3, approximations of the tail probabilities can be derived using complex contour integration and residue theory. The procedure to find the corresponding probability mass function of a pgf, frequently used in the following of this section, is outlined in Appendix 1.

In order to determine the asymptotic behavior of the tail distribution, the dominant singularity of the respective generating functions is important. In e.g. Bruneel (1994) (wherein a single-class ATM queue with a FIFO scheduling discipline is analyzed), it is proven that the dominant singularity lies on the positive real axis and is larger than 1.

First we concentrate on the system contents of class-1. Provided that the pgf's $A_1(z)$ and $S_1(z)$ exhibit no long-tail behavior, which is assumed to be the case here, the dominant singularity z_H of $U_1(z)$ is a zero of $z - S_1(A_1(z))$ and this singularity is a single pole. In the neighbourhood of this pole, we can approximate $U_1(z)$ by

$$U_1(z) \approx \frac{K_1}{z_H - z}, \tag{17}$$

where K_1 can be found by substituting $z = z_H$ in (17). Using residue theory, the tail probability is easily found to yield

$$\text{Prob}[u_1 = n] \approx (1 - \rho_1)\frac{z_H - 1}{S_1'(A_1(z_H))A_1'(z_H) - 1} z_H^{-n}, \tag{18}$$

for large enough n.

The tail behavior of the system contents of class-2 packets is a bit more involved, since it is not a priori clear what the dominant singularity is of $U_2(z)$. This is due to the occurence of the function $Y(z)$ in (9), which is only implicitly defined. First we take a closer look at that function $Y(z)$. The first derivative of $Y(z)$ is given by

$$Y'(z) = \frac{S_1'(A(Y(z),z))A^{(2)}(Y(z),z)}{1 - S_1'(A(Y(z),z))A^{(1)}(Y(z),z)}, \tag{19}$$

with $A^{(j)}(y,z) \triangleq \left.\frac{\partial A(z_1,z_2)}{\partial z_j}\right|_{z_1=y,z_2=z}$ $(j=1,2)$. Consequently, $Y(z)$ has a singularity, denoted as z_B, where the denominator of $Y'(z)$ becomes 0, i.e., $S_1'(A(Y(z_B),z_B))A^{(1)}(Y(z_B),z_B) = 1$. Since $Y(z)$ remains finite in the neighbourhood of z_B, this singularity is not a simple pole. Applying the results from Drmota (1997) one can show that in the neighbourhood of z_B, $Y(z)$ is approximately given by

$$Y(z) \approx Y(z_B) - K_Y(z_B - z)^{1/2}, \tag{20}$$

with $K_Y = \sqrt{\frac{2A^{(2)}(Y(z_B),z_B)}{S_1''(A(Y(z_B),z_B))(A^{(1)}(Y(z_B),z_B))^3 + A^{(11)}(Y(z_B),z_B)}}$, which can be found by taking the limit $z \to z_B$ of (20). $A^{(ij)}(y,z)$ is defined as $\left.\frac{\partial^2 A(z_1,z_2)}{\partial z_i \partial z_j}\right|_{z_1=y,z_2=z}$ (for $i,j=1,2$). From equation (20) it becomes obvious that z_B is a square-root branch point of $Y(z)$. $Y(z)$ has thus two real solutions when $z < z_B$ (the solution we are interested in is the one where $Y(z) < 1$, if $z < 1$), which coincide at z_B, and has no real solution when $z > z_B$. z_B is then of course also a branch point of $U_2(z)$. A second potential singularity z_L of $U_2(z)$ on the real axis is given by the positive zero of the denominator which is a zero of $z - A(Y(z),z)(\bar{\beta}_2 + \beta_2 z)$.

The tail behavior of the system contents of class-2 packets is thus characterized by z_L or z_B, depending on which is the dominant (i.e., smallest) singularity. Three types of tail behavior may thus occur, namely when $z_L < z_B$, $z_L = z_B$ and when z_L does not exist. In those three cases, $U_2(z)$ can be approximated in the neighbourhood of its dominant singularity by:

$$U_2(z) \approx \begin{cases} \dfrac{K_2^{(1)}}{z_L - z} & \text{if } z_L < z_B \\[2ex] \dfrac{K_2^{(2)}}{\sqrt{z_B - z}} & \text{if } z_L = z_B \\[2ex] U_2(z_B) - K_2^{(3)}\sqrt{z_B - z} & \text{if } z_L \text{ does not exist,} \end{cases}$$

where the constants $K_2^{(i)}$ $(i = 1, 2, 3)$ can be found by investigating the behavior of $U_2(z)$ in the neighbourhood of this dominant singularity. Using residue theory, we find the tail probabilities for the three possible cases:

$$\text{(21)} \quad \text{Prob}[u_2 = n] \approx \begin{cases} (1-\rho_T)\dfrac{\bar{\beta}_2 A_2(z_L)(z_L - 1)(A(Y(z_L), z_L) - 1)z_L^{-n}}{z_L(A_2(z_L) - 1)Q_1(z_L)} \\[2ex] \dfrac{1-\rho_T}{K_Y}\sqrt{\dfrac{1}{z_B \pi}}\dfrac{\bar{\beta}_2 A_2(z_B)(z_B - 1)(A(Y(z_B), z_B) - 1)n^{-1/2}z_B^{-n}}{(A_2(z_B) - 1)(\bar{\beta}_2 + \beta_2 z_B)A^{(1)}(Y(z_B), z_B)} \\[2ex] \dfrac{(1-\rho_T)K_Y}{2}\sqrt{\dfrac{z_B}{\pi}}\dfrac{\bar{\beta}_2^2 A_2(z_B)(z_B - 1)^2 A^{(1)}(Y(z_B), z_B)n^{-3/2}z_B^{-n}}{(A_2(z_B) - 1)(z_B - A(Y(z_B), z_B)(\bar{\beta}_2 + \beta_2 z_B))^2}, \end{cases}$$

for large enough n, if $z_L < z_B$, if $z_L = z_B$, and if z_L does not exist respectively. $Q_1(z)$ is defined as

$$Q_1(z) \quad = \quad (A^{(1)}(Y(z), z)Y'(z) + A^{(2)}(Y(z), z))(\bar{\beta}_2 + \beta_2 z) + \beta_2 A(Y(z), z) - 1.$$

The first expression of (21) constitutes a typical geometric tail behavior, the third expression is a typical non-geometric tail behavior and the second expression gives a transition between both. The latter two expressions are found from the approximations of the generating functions by using the Theorem from Appendix 2, which is a theorem proved in Flajolet (1990) using contour integrations (i.e., an extension of the technique proposed in Appendix 1).

Let us now consider the packet delay. The dominant singularity of $D_1(z)$ is a zero of $z - A_1(S_1(z))$, denoted by $\hat{z}_H$, and we can thus approximate the tail behavior of the delay of class-1 packets by

$$\text{Prob}[d_1 = n] \approx \frac{1-\rho_1}{\lambda_1} \frac{S_1(\hat{z}_H)(\hat{z}_H - 1)^2}{z_H(S_1(\hat{z}_H) - 1)(A_1'(S_1(\hat{z}_H))S_1'(\hat{z}_H) - 1)} \hat{z}_H^{-n}, \tag{22}$$

for large enough n. The calculation of these tail probabilities is similar as the calculation of (18). The tail behavior of the delay of class-2 packets is again a bit more involved because of the appearance of the function $V_1(z)$ (and $V_2(z)$) in (12), which is only implicitly known. The first derivative of $V_1(z)$ is given by

$$V_1'(z) = \frac{S_1'(zA_1(V_1(z)))A_1(V_1(z))}{1 - zS_1'(zA_1(V_1(z)))A_1'(V_1(z))}, \tag{23}$$

which, similar as before, indicates that $V_1(z)$ also has a square root branch point $\hat{z}_B$, with $\hat{z}_B S_1'(\hat{z}_B A_1(V_1(\hat{z}_B)))A_1'(V_1(\hat{z}_B)) = 1$. In the neighbourhood of $\hat{z}_B$, $V_1(z)$ is approximately given by

$$V_1(z) \approx V_1(\hat{z}_B) - K_V \sqrt{\hat{z}_B - z}, \tag{24}$$

with $K_V = \sqrt{\dfrac{2A_1(V_1(\hat{z}_B))}{\hat{z}_B[\hat{z}_B^2(A_1'(V_1(\hat{z}_B)))^3 S_1''(\hat{z}_B A_1(V_1(\hat{z}_B))) + A_1''(V_1(\hat{z}_B))]}}$. A second singularity of $D_2(z)$ is given by the dominant zero $\hat{z}_L$ of $zA_1(V_1(z)) - A(V_1(z), V_2(z))$ on the real axis.

So, $D_2(z)$ can be approximated in the neighbourhood of his dominant singularity by:

$$D_2(z) \approx \begin{cases} \dfrac{\hat{K}_2^{(1)}}{\hat{z}_L - z} & \text{if } \hat{z}_L < \hat{z}_B \\[2ex] \dfrac{\hat{K}_2^{(2)}}{\sqrt{\hat{z}_B - z}} & \text{if } \hat{z}_L = \hat{z}_B \\[2ex] D_2(\hat{z}_B) - \hat{K}_2^{(3)}\sqrt{\hat{z}_B - z} & \text{if } \hat{z}_L \text{ does not exist,} \end{cases}$$

where the constants $\hat{K}_2^{(i)}$ $(i = 1, 2, 3)$ can be found by investigating $D_2(z)$ in the neighbourhood of its dominant singularity. By using residue theory once again, the asymptotic behavior of

$D_2(z)$ is given by

$$
(25) \quad \mathrm{Prob}[d_2 = n] \approx \begin{cases} \dfrac{(1-\rho_T)A_1(V_1(\hat{z}_L))(\hat{z}_L - 1)\hat{z}_L^{-n}/\rho_2}{\left.\dfrac{dA(V_1(z),V_2(z))}{dz}\right|_{z=\hat{z}_L} - A_1(V_1(\hat{z}_L)) - \hat{z}_L A_1'(V_1(\hat{z}_L))V_1'(\hat{z}_L)} \\ \dfrac{(1-\rho_T)(\hat{z}_B - 1)A_1(V_1(\hat{z}_B))n^{-1/2}\hat{z}_B^{-n}}{\rho_2 K_V \sqrt{\pi/\hat{z}_B}(Q_2(\hat{z}_B) - \hat{z}_B A_1'(V_1(\hat{z}_B)))} \\ \dfrac{(1-\rho_T)K_V}{2\rho_2\sqrt{\pi/\hat{z}_B}} \dfrac{\hat{z}_B(\hat{z}_B - 1)Q_3(\hat{z}_B)n^{-3/2}\hat{z}_B^{-n}}{(\hat{z}_B A_1(V_1(\hat{z}_B)) - A(V_1(\hat{z}_B),V_2(\hat{z}_B)))^2}, \end{cases}
$$

if $\hat{z}_L < \hat{z}_B$, if $\hat{z}_L = \hat{z}_B$, and if $\hat{z}_L$ does not exist respectively. $Q_j(z)$ $(j = 2, 3)$ are defined as follows:

$$
\begin{aligned}
Q_2(z) &= A^{(1)}(V_1(z), V_2(z)) + \frac{\bar{\beta}_2 z A_1'(V_1(z))A^{(2)}(V_1(z), V_2(z))}{(1 - \beta_2 z A_1(V_1(z)))^2} \\
Q_3(z) &= A_1(V_1(z))Q_2(z) - A(V_1(z), V_2(z))A_1'(V_1(z))
\end{aligned}
$$

The first expression of (25) has a typical geometric tail behavior, the third expression has a typical non-geometric tail behavior and the second expression gives a transition between both.

A quantity of practical interest is the probability that a packet has a delay that exceeds a bound D. We find

$$
(26) \qquad \mathrm{Prob}[d_1 > D] \approx \frac{\mathrm{Prob}[d_1 = D+1]\hat{z}_H}{\hat{z}_H - 1},
$$

for the probability that the delay of a class-1 packet is larger than a bound D. This can be found by summing equation (22) for all appropriate values of n. Analogously, we can calculate the probability that a class-2 packet exceeds a bound D by summing equation (25) for the appropriate values of n. We find

$$\text{Prob}[d_2 > D] \approx \begin{cases} \dfrac{\text{Prob}[d_2 = D+1]\hat{z}_L}{\hat{z}_L - 1} & \text{if } \hat{z}_L < \hat{z}_B \\ \dfrac{\text{Prob}[d_2 = D+1]\hat{z}_B}{\hat{z}_B - 1} & \text{if } \hat{z}_L = \hat{z}_B \\ \dfrac{\text{Prob}[d_2 = D+1]\hat{z}_B}{\hat{z}_B - 1} & \text{if } \hat{z}_L \text{ does not exist,} \end{cases}$$

where we used the approximation that $\sum_{n=D+1}^{\infty} n^{-a} z^{-n} \approx (D+1)^{-a} \sum_{n=D+1}^{\infty} z^{-n}$, with $a = 1/2$ or $3/2$ and which holds for large enough D. Some similar expressions can be found for the probability that the system contents exceeds a certain bound.

Since the results obtained in this section are approximate (due to the dominant pole approximative method), the question remains if the expressions are accurate. From the analysis in Bruneel (1994), it follows that the approximation of the tail probabilities, obtained through the dominant pole method, are better when the dominant pole is more dominant (i.e., the higher the moduli of the other poles compared to the modulus of the dominant pole, the better the quality of the approximation) and when we go further in the tail of the distribution (i.e., coefficient n in expressions (18), (21)-(22) and (25) is higher). We will show in section 6 that the approximate results for the tail probabilities obtained in this section are more than satisfactory.

6 Application: a packet switch

In this section, we apply the obtained results to the special case of an output-queueing packet switch (see Figure 1). We assume two types of traffic. Traffic of class-1 is delay-sensitive (for instance voice) and traffic of class-2 is assumed to be delay-insensitive (for instance data). We investigate the effect of a preemptive resume priority scheduling discipline, as presented in the former of this paper.

The packet arrivals on each inlet are assumed to be i.i.d., and generated by a Bernoulli process with arrival rate λ_T. An arriving packet is assumed to be of class-j with probability

λ_j/λ_T $(j = 1, 2)$ $(\lambda_1 + \lambda_2 = \lambda_T)$. The incoming packets are then routed to the output queue corresponding to their destination, in an independent and uniform way. Therefore, the output queues behave identically and we can concentrate on the analysis of 1 output queue. In view of the previous, the arrivals of both types of packets to an output queue are generated according to a two-dimensional binomial process. It is fully characterized by the following joint pgf

$$A(z_1, z_2) = (1 - \frac{\lambda_1}{N}(1 - z_1) - \frac{\lambda_2}{N}(1 - z_2))^N. \tag{27}$$

Notice that if $N \to \infty$, the arrival process becomes a superposition of two independent Poisson streams. In the remainder of this section, we assume that $N = 16$.

[Figure 1 about here.]

In Figures 2 and 3, the mean and variance of the system contents of class-1 and class-2 packets is shown as a function of the total load ρ_T, when the service times of class-1 packets are deterministically equal to 2 $(\mu_1 = 2)$ and $\beta_2 = 0.5$ $(\mu_2 = 2)$. The fraction of the class-1 load in the total load, denoted by α, is 0.25, 0.5 and 0.75 respectively. We clearly see the influence of the priority scheduling discipline. The mean and variance of the system contents of class-1 packets remains low, even if the fraction of class-1 packets is high. The mean value and variance of the system contents of class-2 packets on the other hand is high, especially when the system is heavily loaded.

[Figure 2 about here.]

[Figure 3 about here.]

In Figures 4 and 5, the mean value and variance of the packet delay of class-1 and class-2 packets is shown as a function of the total load ρ_T, when the service times of both classes are geometrically distributed with parameter 0.5, i.e., $\mu_j = 2$ $(j = 1, 2)$ and $\alpha = 0.25, 0.5$ and 0.75 respectively. In order to compare with FIFO scheduling, we have also shown the mean value and variance of the packet delay in that case. Since, in this example, the service times of the class-1 and class-2 packets are equally distributed, the packet delay is then of course the same for class-1 and class-2 packets, and can thus be calculated as if there is only one class of packets arriving according to an arrival process with pgf $A_T(z)$. This has already been analyzed, e.g., in Bruneel (1993a). The influence of the priority scheduling discipline on the packet delay becomes obvious from these figures: mean and variance of the delay of class-1 packets reduces significantly. The price to pay is of course a higher mean and variance of the delay of class-2 packets. If this kind of traffic is not delay-sensitive, as assumed, this is not a too big a problem. Also, the smaller the fraction of high-priority packets in the overall traffic mix, the lower the mean and variance of the packet delay of both classes will be.

[Figure 4 about here.]

[Figure 5 about here.]

Figure 6 (Figure 7 respectively) shows the mean delay of high- and low-priority packets when the service times of high-priority packets are deterministic, as a function of the mean service time of the low-priority packets (high-priority packets respectively), i.e., μ_2 (μ_1 respectively), when $\mu_1 = 2$ ($\mu_2 = 2$ respectively), $\rho_T = 0.75$ and $\alpha = 0.25, 0.5$ and 0.75 respectively. The figures show that the mean packet delay of high-priority packets is not influenced by the mean service time of class-2 packets, while it is proportionally increasing with the mean service time of class-1 packets (when the load of high- and low-priority packets is kept constant). The mean packet delay of class-2 packets on the other hand is proportionally increasing with the mean service time of class-2 packets (Figure 6) and with the mean service time of class-1 packets (Figure 7). Because of the preemptive priority scheduling, the mean delay of high-priority packets is only influenced by its own arrival and service process, while the mean delay of low-priority packets is influenced by the arrival and service processes of both classes.

[Figure 6 about here.]

[Figure 7 about here.]

In the next figures, we are going to illustrate the tail behavior of the packet delay. The tail behavior of the system contents is similar (see section 5) and as a result similar plots as the ones for the packet delay can be constructed (but are omitted here). We have shown in section 5, that the tails of class-2 packet delay can have 3 types of behavior, depending on which singularity of $D_2(z)$ is dominant. In case of the output queueing switch considered in this section, Figures 8 and 9 show for which combination of class-1 and class-2 loads the transition type behavior occurs for the packet delay, i.e., for which combination of loads the regular pole and the branch point coincide, for several values of β_2. In Figure 8, the service times of class-1 packets are deterministically equal to 2, while in 9 the service times of class-1 packets are geometrically distributed with mean 2. In the region above the curves, the tail behavior is geometric for the respective ρ_1 and ρ_2, while below the curves the tail behavior is typically non-geometric. Note that in the area above the line defined by $\rho_1 + \rho_2 = 1$ in Figures 8 and 9, the total load is larger than 1, and as a result, the system becomes unstable. As can be seen, the higher β_2 the smaller the region where the tail behavior is non-geometric. By comparing both figures, we see that the transition between geometric and non-geometric tails highly depends on the service time distribution of the high-priority packets.

[Figure 8 about here.]

[Figure 9 about here.]

Figure 10 shows the tail behavior of the packet delay of class-1 and class-2 packets if $\rho_1 = 0.4$, $\mu_1 = 2$, $\beta_2 = 0.25$ and $\rho_2 = 0.05$ (non-geometric behavior), approximately 0.22 (transition type behavior) and 0.4 (geometric behavior) respectively. Tail behavior of packet delay of class-1 packets is of course the same for the three cases, since the arrival process of class-1 packets is identical in all cases, and class-2 packets are 'invisible' for the high-priority class-1 packets due to the preemptive service discipline. We have also compared our approximations with simulation results (marks in the figures). The figures show that the approximations for the tail probabilities of the delay of both classes is very good.

[Figure 10 about here.]

To conclude this section, we analyse the following case-study. Consider two traffic classes generating packets that arrive in a common multiplexer buffer where they are temporarily stored before transmission. The packet arrival process of both classes is described by a joint pgf given by expression (27). The mean service time of both classes is equal to 2 (service times of packets of the high-priority class are deterministic, service times of low-priority packets are geometrically distributed). For both classes, their respective packet delay must satisfy the constraint $\text{Prob}[d_j > T_j] < 10^{-X_j}$, i.e., the fraction of packets of class-j that have a delay larger than the treshold T_j may not exceed 10^{-X_j}, where T_j and X_j depend on the application under consideration. It is assumed that class-1 packets are delay-sensitive, implying that they are given priority over class-2 packets (and $T_1 < T_2$, since it makes no sense to have a higher delay treshold for delay-sensitive traffic). Class-2 traffic may be loss-sensitive, and the amount of packets that is rejected due to their delay treshold being exceeded must be sufficiently small. Therefore, in the remainder we will set $X_2 = 9$ and $X_1 \equiv X$ (where the latter may be varied). It is clear that the performance of both traffic classes, in particular their delay characteristics, can be studied using the results derived throughout this paper.

The question we wish to answer is the following: what is the maximal load (denoted by $\rho_{T,max}$), as a function of the traffic mix α, that still fulfils the two constraints? In Figure 11, we show the maximal load as a funtion of α when $T_1 = 10$, $T_2 = 100$ and several values of X. The constraint for the delay of class-2 packets is the same for all X, i.e., $\text{Prob}[d_2 > 100] < 10^{-9}$. For $X < 4$, we see that this constraint is the decisive one. We notice that the maximal load shows a discrepancy when α reaches approximately 0.8. At this point, the tail behavior for the low-priority packets changes from geometric to non-geometric tail behavior. The sudden change near 0.8 is probably due to the lack of accurateness in the tail behavior of the class-2 delay near the transition. Near this value for α, the maximal load we find is thus not that accurate, but one can see that the incorrectness is in the order of a few percentages. For higher X, the constraint for the delay of the high-priority traffic becomes decisive for high α, i.e. when more class-1 packets arrive. In Figure 12, we show $\rho_{T,max}$ as a funtion of α when $X = 5$, $T_2 = 100$ and $T_1 \geq 6$. The constraint for the delay of class-2 packets is again the same for all T_1. For $T_1 > 13$, we see that this constraint is the decisive one. For lower T_1, the constraint for the delay of the high-priority traffic becomes decisive for high α, i.e., when more class-1 packets arrive. Finally, in Figure 13, the maximum load as a function of α is shown, when $X = 3$, $T_1 = 10$ and several values of T_2. For low T_2, the constraint for the low-priority traffic is always the most stringent, while for $T_2 \geq 150$, the constraint for the high-priority traffic is decisive for high α.

The behavior depicted in these three figures can be explained as follows. For $\alpha = 0$, the traffic mix consists of low-priority packets only, and $\rho_{T,max}$ is relatively high, depending on the value of T_2. As α increases, $\rho_{T,max}$ gradually decreases (but is still determined by T_2) since the growing fraction of high-priority packets causes the mean low-priority packet delay to rise. Then, as α further increases, a transition point is reached, which is defined as the value of α and ρ_T for which $\text{Prob}[d_1 > T_1] = 10^{-X}$ and $\text{Prob}[d_2 > T_2] = 10^{-9}$. Beyond this transition point, the bound set by T_1 becomes predominant, and $\rho_{T,max}$ further decreases due to the ever increasing presence of high-priority packets in the traffic mix. These figures show that the maximum allowable load can strongly depend on the delay boundaries T_1 and T_2 set on the high- and low-priority packet delays, and the traffic mix α.

[Figure 11 about here.]

[Figure 12 about here.]

[Figure 13 about here.]

7 Conclusion

In this paper, we have analyzed a discrete-time queue with a preemptive resume priority scheduling discipline and two priority classes. Service times of the high- and low-priority packets are generally and geometrically distributed respectively. We have derived the steady-state joint generating function of the system contents of both classes and the generating functions of the delay of both classes. The expressions for these pgf's are not explicit, but we have proven that the moments of the distributions can be found explicitly in terms of the system parameters. Furthermore, we have shown that approximate tail probabilities of system contents and packet delay can be calculated and that the tail probabilities of the low-priority system contents and packet delay not necessarily have an exponential decay. Using the expressions of the moments and tail distributions, we have discussed the impact of the priority scheduling discipline on the performance characteristics in the special case of an output-queueing packet switch.

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Appendix 1 : Calculation of the Probability Mass Function

Given a generating function $X(z) \triangleq \sum_{n=0}^{\infty} x(n)z^n$, the question is how to find an explicit, practically usable expression for its corresponding pmf $x(n)$. From the definition of $X(z)$ it follows that $x(n)$ is the coefficient of z^n in the expansion of $X(z)$ about $z = 0$, or equivalently the coefficient of z^{-1} in the expansion of $z^{-1-n}X(z)$ about $z = 0$. $x(n)$ is thus by definition the residue of the function $z^{-1-n}X(z)$ in the point $z = 0$. Since the multiplicity of the pole $z = 0$ of $z^{-1-n}X(z)$ depends linearly on n, calculating the residue in $z = 0$ is nearly impossible for large n (since evaluating the residue in an k-multiple pole requires k derivations). Using the residue theorem of Cauchy however, it is proven that

$$\begin{aligned} x(n) &= \mathrm{Res}_{z=0}[X(z)z^{-1-n}] \\ &= \frac{1}{2\pi i}\oint\limits_{C_1} X(z)z^{-1-n}dz - \sum_{j=0}^{m} \mathrm{Res}_{z=z_j} X(z)z^{-1-n} \end{aligned}$$

with $i = \sqrt{-1}$, C_1 a contour with infinite radius and z_j the poles of $X(z)$. The contour integral in the former expression is normally easy to calculate (in most cases the term equals zero). The sum of residues can be approximated by the residue in the dominant pole of $X(z)$. As a result, an easy, practically usable formula to calculate approximate tail probabilities is obtained.

Appendix 2 : Inversion of $(1-z)^\alpha$

Theorem 1 *Assume that, with the sole exception of the singularity* $z = 1$,

$$F(z) \triangleq \sum_{n=1}^{\infty} f(n)z^n,$$

is analytic in the domain

$$\Delta = \{z : |z| \leq 1 + \eta, |Arg(z-1)| \geq \theta\} \setminus \{1\},$$

in which η *is a positive real number and* $0 < \theta < \pi/2$. *Assume further that as* z *tends to 1 in* Δ,

$$F(z) = K(1-z)^\alpha,$$

with $\alpha \notin \mathbb{N}$. *Then, as* $n \to \infty$,

$$f(n) = \frac{K}{\Gamma(-\alpha)} n^{-\alpha-1}.$$

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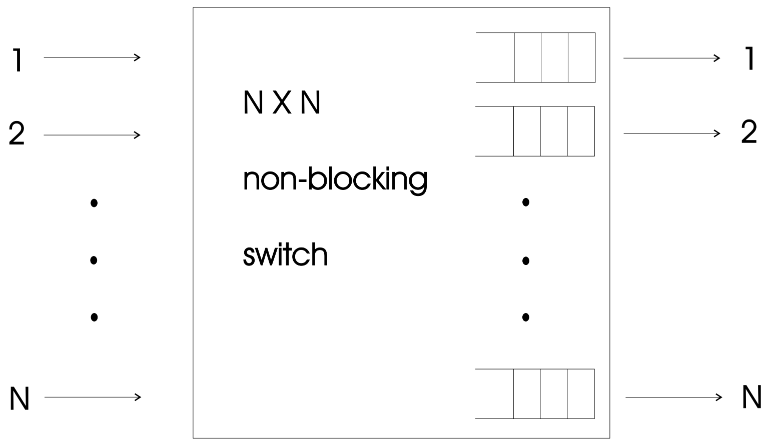
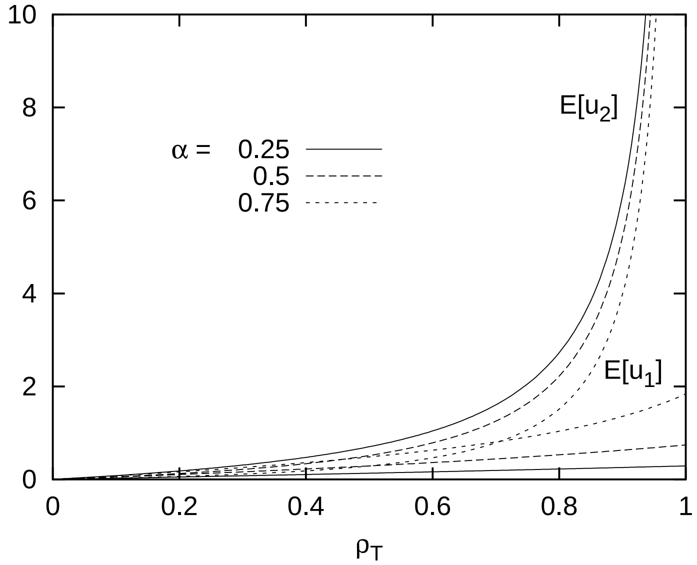


Figure 1: An output-queueing packet switch



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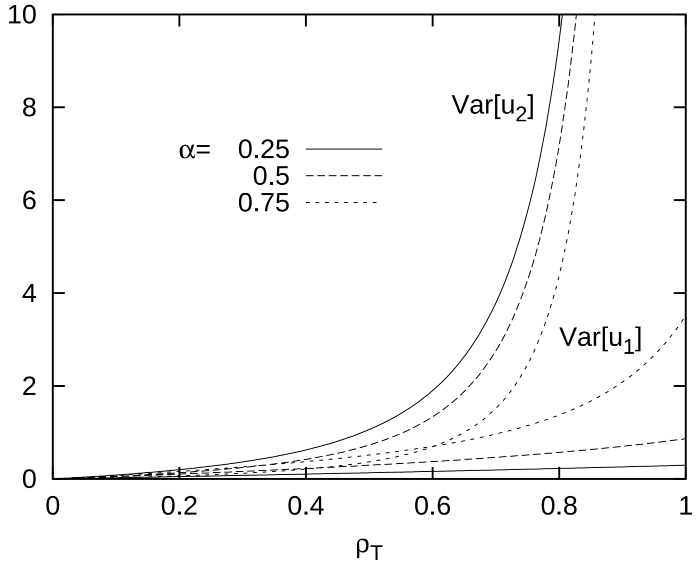

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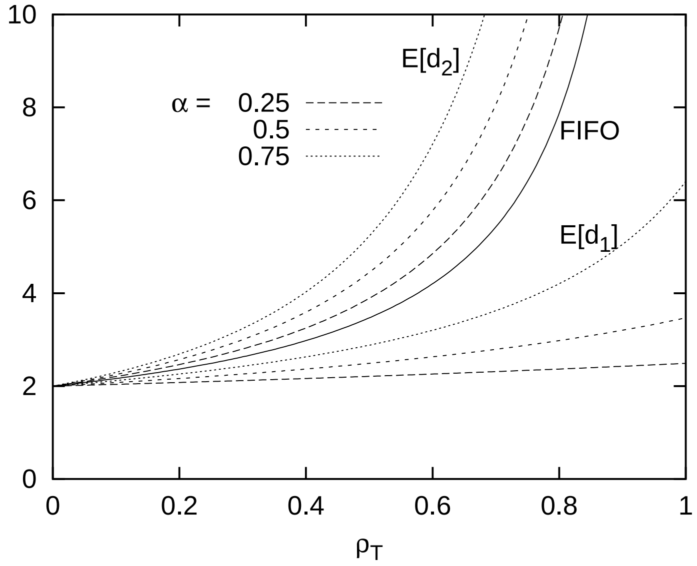

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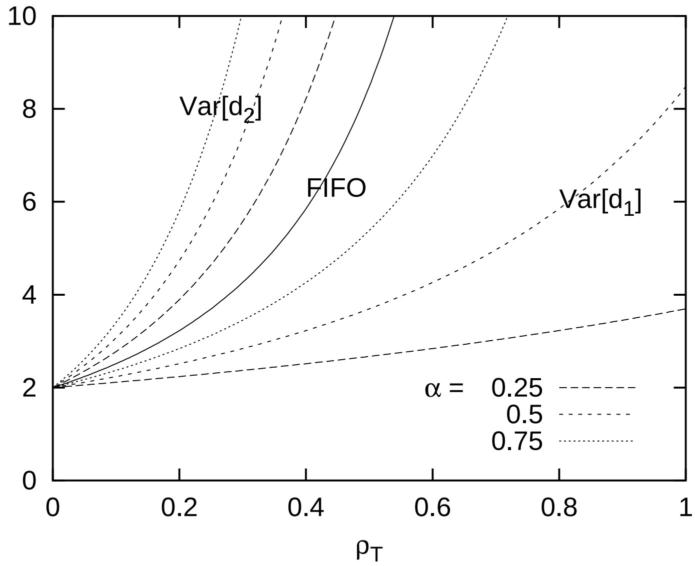

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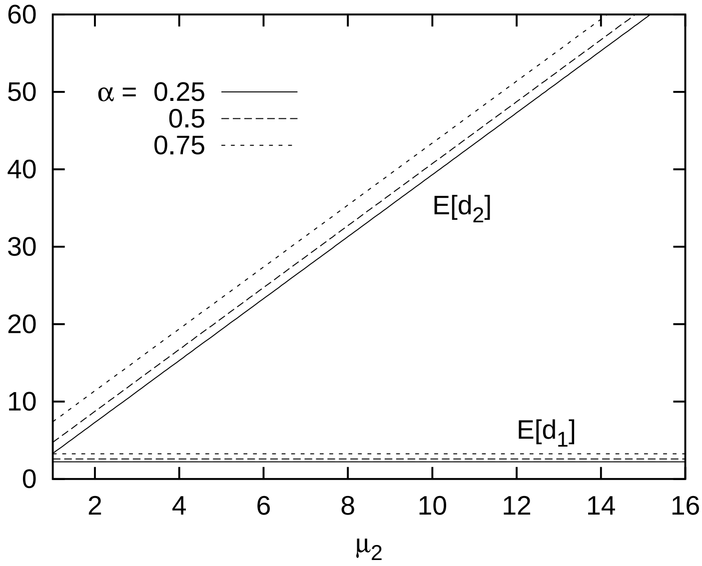

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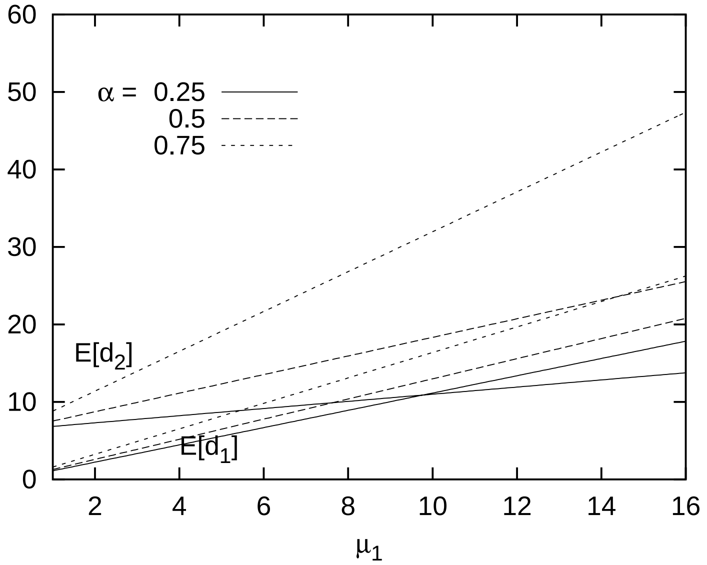

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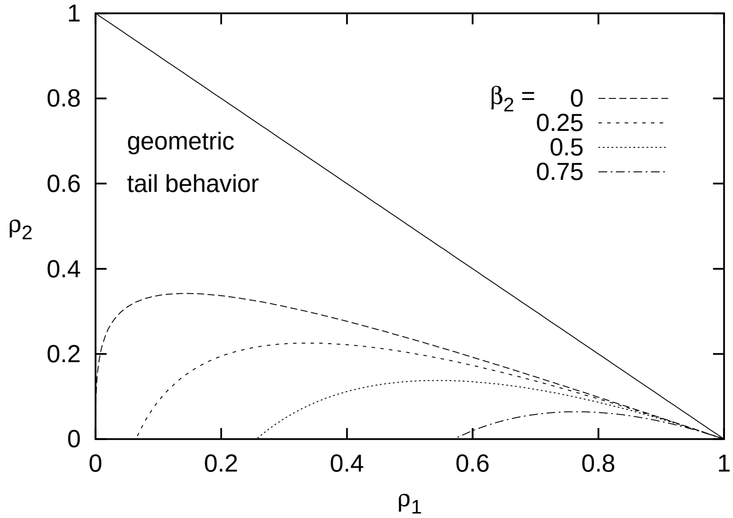

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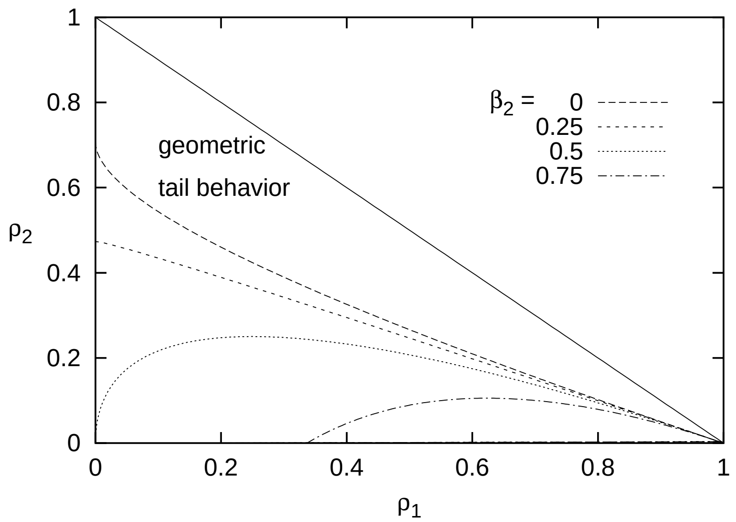

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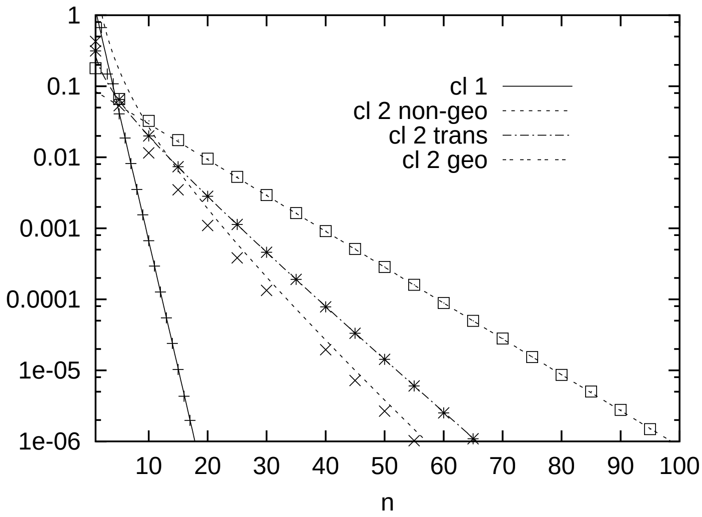

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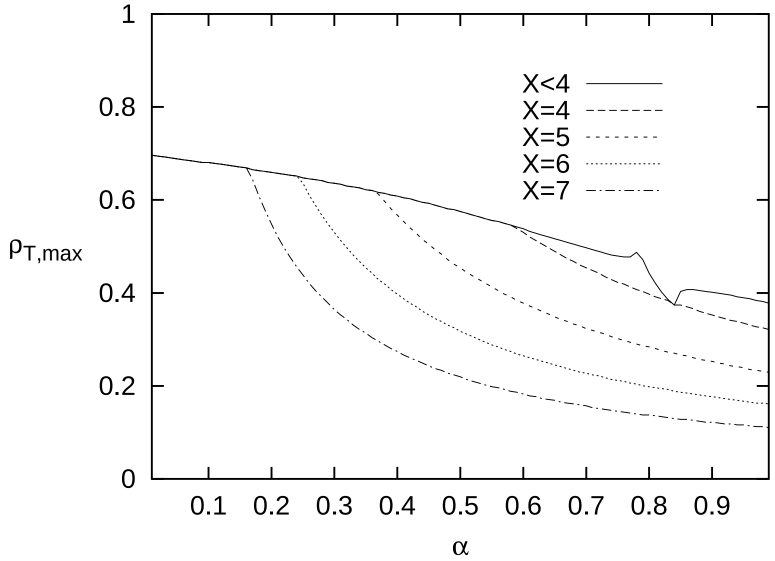

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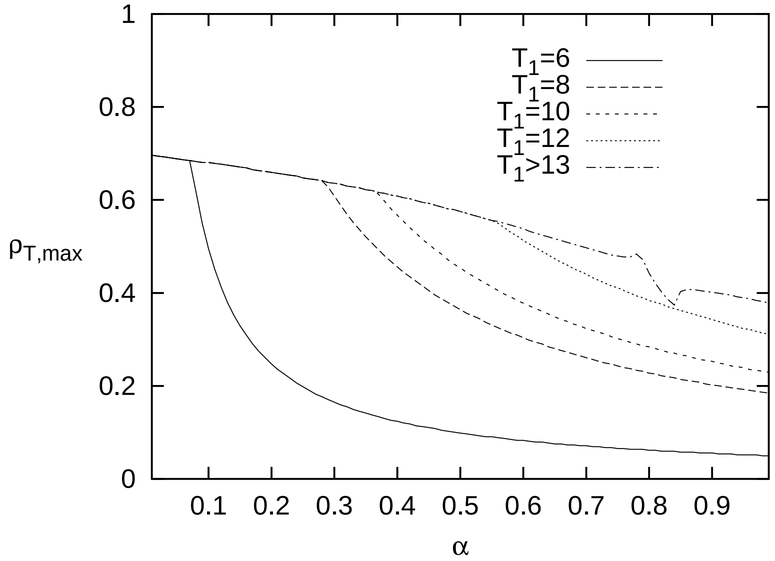

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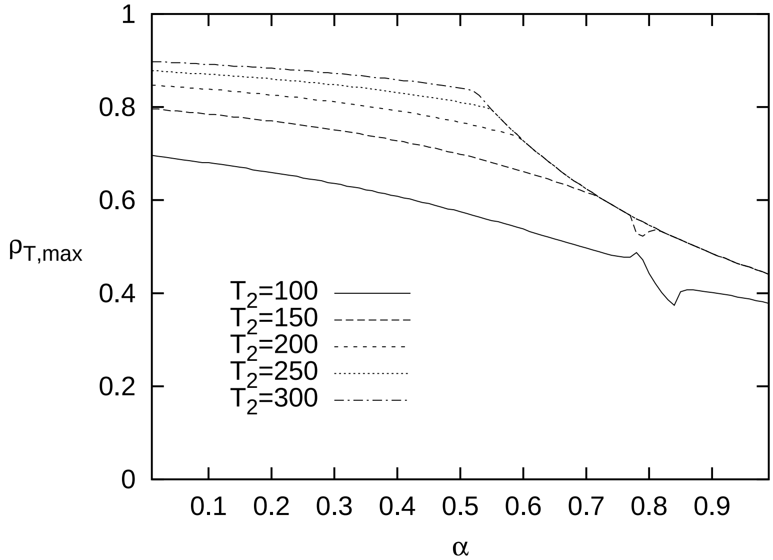

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